Equilibria in Repeated Games with Endogenous Separation

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Abstract

Games with endogenous separation are repeated games in which the set of choices that a player can make after every stage game includes the option to leave the current partnership and keep on playing in a newly-formed partnership. In the setting of population games, we present a general framework to analyze equilibria in games with endogenous separation, focusing on neutral stability in the unrestricted strategy space. We introduce path-protecting strategies and provide a constructive proof of existence. *JEL* classification numbers: C72, C73.

Keywords: Endogenous separation; neutral stability; path-protecting strategy; repeated games

1 Introduction

Games with endogenous separation[1] are repeated games (Mailath and Samuelson, 2006) in which the set of choices that a player can make after every stage game includes the option to leave the current partnership and keep on playing in a newly-formed partnership. Partnerships may be broken because some players decide to break them (endogenous separation) or for reasons that do not depend on the players' choices (exogenous separation). In most models, one single player's decision to leave is sufficient to break the partnership, but other alternatives have also been considered.[2] We assume that there is no information flow between partnerships (Ghosh and Ray, 1996), so there are no reputation effects: single players (those who make up new partnerships) are anonymous.[3]

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Abbreviations.

[1] Games with endogenous separation are also known as *voluntarily separable repeated games* (Fujiwara-Greve and Okuno-Fujiwara, 2009; Fujiwara-Greve et al., 2012, 2015), *voluntary partnership games* (Vesely and Yang, 2010), *games with conditional dissociation* (Izquierdo et al., 2010, 2014), *voluntary continuation games* (Vesely and Yang, 2013) or games with *endogenous match separation* (Deb et al., 2020). Newton (2018) provides a nice short summary, in the context of assortativity (Nax and Rigos, 2016).

[2] See e.g. Kurokawa (2022) or Křivan and Cressman (2020).

[3] Fujiwara-Greve et al. (2012) consider a model where players may voluntarily provide information across partnerships in the context of the Prisoner's Dilemma.

To our knowledge, there is still no standard framework for the analysis of games with endogenous separation in the unrestricted strategy space, not even for the two-player case. Some proposed approaches that we draw on can be found in Carmichael and MacLeod (1997) and Fujiwara-Greve and Okuno-Fujiwara (2009), which focus on the Prisoner's Dilemma, as well as in Vesely and Yang (2010) and Izquierdo et al. (2021).[4] Here we present a framework that extends the setting of evolutionary population games (Sandholm, 2010) to symmetric two-player games with endogenous separation.

Within this framework, we discuss different equilibrium concepts. While some equilibrium concepts, such as Nash equilibrium, can be easily extended to games with endogenous separation, the extension of stronger equilibrium concepts such as neutral stability is not so straightforward. One difficulty is associated with finding a convenient characterization of the payoff function for a group of potential invaders in a population. Then, there are different standard definitions of neutral stability (Bomze and Weibull, 1995). Most of those definitions are equivalent in the standard model of two-player games, where payoff functions are linear, but they are not equivalent (and some of them do not guarantee desirable dynamic stability properties) when payoff functions are not linear (Bomze and Weibull, 1995), which is the case in games with endogenous separation. It remained a challenge (see appendix A) to find a definition of neutral stability for games with endogenous separation that is i) simple and consistent with some standard definition of neutral stability, and ii) guarantees desirable dynamic stability properties, such as Lyapunov stability in the replicator dynamics.

Here, after a convenient definition and characterization of payoff functions, we propose a definition of neutral stability that is a direct adaptation for population games of the original concept (Maynard Smith, 1982; Banerjee and Weibull, 2000), and which allows us to bring known dynamic implications of such equilibria to our framework.

We pay special attention to monomorphic equilibrium states, or conventions, at which every player uses the same strategy. Our analysis of neutral stability shows a fundamental limitation of strategies with finite break-up period, i.e., strategies that, when playing against themselves, decide to break up the partnership at some point: such strategies cannot be neutrally stable in most games. We analyze a special type of strategies that we call *path-protecting*. A path-protecting strategy never leaves a partner who mimics its behavior and, if adopted by all the players in a population, guarantees that any player who deviates from the equilibrium path obtains a strictly lower payoff than the population's average. The concept of path-protecting strategy generalizes the idea of *trust-building* strategies that appear in previous works focused on the Prisoner's Dilemma or games with a similar structure (Datta, 1996; Ghosh and Ray, 1996; Carmichael and MacLeod, 1997; Kranton, 1996; Fujiwara-Greve and Okuno-Fujiwara, 2009). Path-protecting strategies will be shown to be neutrally stable, and Lyapunov stable in the replicator dynamics.

The analysis of existence of path-protecting strategies will lead to a folk theorem

[4] Several studies focus on specific sets of strategies, such as Aktipis (2004, 2011) and Premo and Brown (2019) in a spatial setting, Izquierdo et al. (2010, 2014), Zheng et al. (2017) and Li and Lessard (2021). For experimental studies see Zhang et al. (2016) and the references therein.

for neutral stability, which shows that in a game with endogenous separation, for large enough values of the continuation probability, any payoff between the pure minmax payoff and the maximum symmetric payoff of the stage game can be obtained as the equilibrium payoff of some neutrally stable strategy.

Path-protecting strategies present some parallelisms and some differences with the classical *trigger* strategies that, in standard repeated games, prevent deviations from an equilibrium path by playing a minmax action after a deviation. In standard repeated games, trigger strategies protect a path by the threat of punishment, but such potential punishment does not materialize in the equilibrium path. In games with endogenous separation, a player who deviates can avoid punishment from a partner by breaking up the partnership, so a convention in a population needs to ensure that players who deviate and go (or are sent) to find a new partner undergo a journey through the wilderness or *deviation-deterring* phase after starting a new partnership. Because of the anonymity assumption, every player when starting a partnership needs to go through the deviation-deterring phase, which has to be part of the equilibrium path.

The concept of path-protecting strategy will also be extended to polymorphic path-protecting equilibrium states.

The paper is structured as follows. In section 2 we define games with endogenous separation derived from normal-form stage games, and we discuss their main elements: strategies, population states, pool states and payoff functions. In section 3 we provide definitions for Nash states and for neutrally stable states in these games. Previous definitions of neutral stability for this framework, such as those by Carmichael and MacLeod (1997), Fujiwara-Greve and Okuno-Fujiwara (2009) or Izquierdo et al. (2021) are different from each other and, in the first two cases, are not consistent with standard definitions of neutral stability (see appendix A). Our approach to define neutral stability starts by characterizing an adequate payoff function for strategies in terms of population states. Once this is done, it becomes natural and straightforward to adapt a standard definition of neutral stability (Banerjee and Weibull, 2000) to repeated games with endogenous separation. We then show that neutral stability thus defined guarantees Lyapunov stability in the replicator dynamics for games with endogenous separation. Section 4 introduces *path-protecting* strategies, and shows how these strategies can constitute monomorphic neutrally stable states for sufficiently high exogenous continuation probabilities. Here we also provide a *folk theorem* for neutrally stable strategies in games with endogenous separation. Section 5 discusses polymorphic neutrally stable states, shows a strong limitation to the existence of polymorphic equilibria made up by different path-protecting strategies, and extends the concept of path-protecting strategy to path-protecting equilibrium state. Finally, section 6 concludes the paper. Most of the proofs can be found in appendix D.

2 Repeated Games with endogenous separation

In this section we present repeated Games with Endogenous Separation derived from normal-form stage games. For simplicity, we focus the presentation and the analysis on symmetric two-player stage games.

We consider a unit-mass population of agents who are matched in couples or *partnerships* to play a symmetric two-player normal-form stage game. The stage game $G = \{A, U\}$ is defined by an action set $A = \{a_1, ..., a_n\}$, and a payoff function $U\colon A^2 \to \mathbb{R}$, where $U(a_k, a_l)$ represents the payoff obtained by a player using action a_k whose opponent plays action a_l. Every stage game G has an associated repeated game with endogenous separation G^{Ends}, which is characterized in this section. Following Mailath and Samuelson (2006), we refer to choices in the stage game G as *actions*, reserving *strategy* for behavior in the repeated game.

2.1 Strategies in G^{Ends}

After playing a stage game G, partnerships may remain together and play the stage game again. A partnership is broken if either one of the players, according to their strategy, decides to break it (endogenous separation) or if some exogenous factor breaks the partnership, which happens with probability $(1 - \delta) \in (0, 1)$ after every interaction (exogenous separation). Thus, δ is the continuation probability of the partnership assuming that both players decide to stay. At the beginning of every (discrete) time period, all single players are randomly (re-)matched in partnerships, and then all players play the stage game, i.e., every player plays the stage game at every period, either in newly-formed partnerships or in older ones. We assume there is no information flow between partnerships.

Considering the sequence of action profiles taken in a partnership, let the stage-t game, with $t \in \{1, 2, ...\}$, be the t^{th} time that the stage game has been played in that partnership, assuming the partnership has not been broken before. A strategy i for a player determines the choices that the player would make given any past history of play within a partnership. If the strategies followed by the two players in a partnership are i and j, the action profile played at stage t (assuming the partnership survives to play for the t^{th} time together), is $a_{ij}^{[t]} \equiv (a_i^{[t]}, a_j^{[t]}) \in A^2$, where $a_i^{[t]}$ is the action played by the player using strategy i (at stage t) and $a_j^{[t]}$ is the action played by the player using strategy j.

Denoting the null (empty) history by $a^{[0]}$, and taking $(A^2)^0 \equiv \{\emptyset\}$, a *history of play of length* $t \geq 0$, $a^{[0,t]} = (a^{[0]}, a^{[1]}, ..., a^{[t]}) \in (A^2)^t$, is a sequence of t action profiles.[5] The set of all possible histories of any length (including the empty history, or history of length 0) is

$$\mathcal{H} \equiv \bigcup_{t=0}^{\infty} (A^2)^t.$$

Let $\tilde{A} \equiv A \cup \{\textit{break}\}$ be the set of choices, where *break* represents the decision to break the current partnership. A strategy i for the repeated game is a mapping $i : \mathcal{H} \to \tilde{A}$, from the set of possible histories to the set of choices, that prescribes one choice $i(a^{[0,t]}) \in \tilde{A}$ for

[5] $a^{[0,t]}$ represents some sequence of t action profiles, while $a_{ij}^{[0,t]}$ represents the first t action profiles generated by strategy i when playing against strategy j, assuming they do not break up before stage t.

every possible history $a^{[0,t]}$, for every $t \geq 0$. As players in a new partnership are assumed to play at least once together before deciding whether to break their partnership, we require $i(\emptyset) \in A$. Let Ω be the set of strategies.

Note that:

- We assume $0 < \delta < 1$. The process for $\delta = 0$, where every partnership is exogenously broken after every stage game, would correspond to the standard framework for evolutionary population games.
- Constraining the strategy space to strategies that never choose *break* provides an evolutionary framework for standard indefinitely repeated games, where the stage game is iteratively repeated with probability δ.

2.2 States and payoffs in G^{Ends}

We consider populations where the number of different strategies being played at any time is finite. Let x_i be the fraction of the population using strategy $i \in \Omega$. A (population) state x is a strategy distribution over Ω with finite support $\mathbb{S}(x) \subset \Omega$, i.e., x is a function from Ω to $[0, 1]$ that:

i) assigns a positive value $x_i > 0$ to each strategy i in a finite set $\mathbb{S}(x)$,

ii) assigns the value 0 to strategies that are not in $\mathbb{S}(x)$, and

iii) satisfies $\sum_{i \in \mathbb{S}(x)} x_i = 1$.

Let $\mathbb{D}$ be the set of distributions with finite support, and let e_i represent the monomorphic state at which all players use strategy i (i.e., the distribution satisfying $x_i = 1$ and $x_j = 0$ for every $j \in \Omega \setminus \{i\}$).

Consider an index $\mathcal{T}$ for periods of play of the game in the population. At every period, single players are matched and every player plays a stage game. In contrast, index t refers to repetitions of the stage game within a partnership: at period $\mathcal{T}$, after matching and before playing the stage game, every partnership has its own value for t, which, if the partnership has just been matched at that period, is set to 0 before playing the stage game and becomes 1 after playing the stage game. For any pair of strategies i and j, let their endogenous break-up period $T_{ij} \geq 1$ be the number of stages that an i-j partnership is to play together if the partnership is not broken by exogenous factors (i.e., the number of stage games they play together before one of them decides to break up). If an i-j partnership never breaks up endogenously, let $T_{ij} = \infty$.

To calculate the average payoff $F_i(x)$ obtained by players using strategy i when the population state is x (average per player in each period), we consider the associated stationary strategy distribution p in the pool of singles, which satisfies at every period $\mathcal{T}$ that:

- Before matching, the mass of players in the pool of singles is a stationary value ϕ. The mass of single i-players in the pool is $\phi \, p_i$.

- After matching, the mass of i-players just matched to j-players, i.e., the mass of i-players in newly-formed (0-period-old) i-j partnerships, is $\phi\, p_i\, p_j$.

- For $1 \leq t \leq T_{ij}$, the mass of i-players in $(t-1)$-period-old i-j partnerships (after matching and before playing), is $\phi\, p_i\, p_j\, \delta^{t-1}$. These are the i-players that were matched in i-j partnerships $(t-1)$ periods ago and have survived exogenous (and endogenous) separation to play their t^{th} stage game in the current period $\mathcal{T}$. The total mass or fraction of i-players in the population is then

$$x_i = \phi \sum_{j \in \mathbb{S}(x)} p_i\, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} = \phi \sum_{j \in \mathbb{S}(x)} p_i\, p_j\, \frac{1-\delta^{T_{ij}}}{1-\delta}$$

and considering that $\sum_{j \in \mathbb{S}(x)} x_j = 1$, we have

$$x_i = \frac{p_i \sum_{j \in \mathbb{S}(x)} p_j (1-\delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(x)} p_k\, p_j (1-\delta^{T_{kj}})}. \tag{1}$$

Equation (1) defines a function $f : \mathbb{D} \to \mathbb{D}$ such that $x = f(p)$, which provides the population state x corresponding to pool state p.

- Let $(a_i^{[t]}, a_j^{[t]}) \in A^2$ be the action profile played at the t^{th} stage of an i-j partnership, with the first action in the profile corresponding to the player using strategy i and the second action in the profile corresponding to the player using strategy j. The total payoff obtained (at each and every period $\mathcal{T}$) by the mass of i-players is

$$\phi \sum_{j \in \mathbb{S}(x)} p_i\, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1}\, U(a_i^{[t]}, a_j^{[t]}),$$

so, dividing by the mass of i-players, we have that the per-period per-player average payoff to an i-player is

$$F_i^p(p) \equiv (1-\delta)\, \frac{\sum_{j \in \mathbb{S}(x)} p_j \sum_{t=1}^{T_{ij}} \delta^{t-1}\, U(a_i^{[t]}, a_j^{[t]})}{\sum_{j \in \mathbb{S}(x)} p_j\, (1-\delta^{T_{ij}})}, \tag{2}$$

which is defined for every $i \in \Omega$.

From (2) we have a formula for $F_i^p(p)$ that provides the payoff to strategy i corresponding to pool state p, and from (1) we have a formula $x = f(p)$, that provides the population state x corresponding to pool state p. In order to use existing results and concepts from the literature in population games, it would be convenient to have payoff functions F_i that provide the payoff to strategy i corresponding to population state x, i.e., $F_i(x)$. Considering $x = f(p)$ as defined in (1), there is an inverse function f^{-1}

such that $p = f^{-1}(x)$ (Izquierdo et al., 2021), so we can define payoff functions F_i from population states as

$$F_i(x) = F_i^p(f^{-1}(x)). \tag{3}$$

Interestingly, for more than three strategies, $f^{-1}(x)$ does not admit a general closed-form algebraic expression (Izquierdo et al., 2021). Our results are based on a series of properties of the payoff functions $F_i(x)$ that we indicate in the following section.

Finally, for a group of players with strategy distribution $y \in \mathbb{D}$ entering a population with strategy distribution x, we define the average *payoff of y against x*, $E(y, x)$, as:

$$E(y, x) \equiv \sum_{i \in \mathbb{S}(y)} y_i F_i(x). \tag{4}$$

We can interpret this payoff as the average payoff obtained by a very small mass of players whose strategy distribution is y (sometimes called *mutants* or *entrants*) when they play in a population of players whose strategy distribution is x.

2.3 Properties of the payoff functions for G^{Ends}

The payoff functions $F_i : \mathbb{D} \to \mathbb{R}$, defined in (3) for every $i \in \Omega$, satisfy the following properties:

- At monomorphic population states (where $x = e_j = p$) we have, from (2):

$$F_i(e_j) = \frac{1-\delta}{1-\delta^{T_{ij}}} \sum_{t=1}^{T_{ij}} \delta^{t-1} U(a_i^{[t]}, a_j^{[t]}). \tag{5}$$

  Note that the payoff $F_i(e_j)$ to an i-player in a population of j-players is a convex combination of the stage payoffs $U(a_i^{[t]}, a_j^{[t]})$ for $1 \leq t \leq T_{ij}$.

- It follows from (2), (3) and (5) that, for $p = f^{-1}(x)$, we have

$$F_i(x) = F_i^p(p) = \sum_{j \in \mathbb{S}(x)} \frac{p_j\,(1-\delta^{T_{ij}})}{\sum_{k \in \mathbb{S}(x)} p_k\,(1-\delta^{T_{ik}})} F_i(e_j), \tag{6}$$

  which shows that $F_i(x)$ is a convex combination of the payoffs $F_i(e_j)$ for $j \in \mathbb{S}(x)$, with (strictly) positive coefficients for the convex combination.

Let the *path* $a_{ij}^{[1,T_{ij}]} = ((a_i^{[1]}, a_j^{[1]}), (a_i^{[2]}, a_j^{[2]}), ..., (a_i^{[T_{ij}]}, a_j^{[T_{ij}]}))$ be the series of T_{ij} action profiles that strategy i generates when playing with strategy j until they decide to break up. Let the *repeated path* $h_{ij}^{[\infty]}$ be the infinite series of action profiles that corresponds to (or is generated by) one i-player in a population of j-players, with no exogenous separation and with re-matching after each endogenous separation:

$$h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]}, a_{ij}^{[1,T_{ij}]}, ...). \tag{7}$$

For a sequence of T action profiles $a^{[1,T]}$, where the t^{th} action profile in the sequence is $a^{[t]} \in A^2$, let

$$V(a^{[1,T]}) \equiv \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} U(a^{[t]}).$$

From the previous definitions and the properties of geometric series, we have:

$$F_i(e_j) = V\left(a_{ij}^{[1,T_{ij}]}\right) = V\left(h_{ij}^{[\infty]}\right) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} U(h_{ij}^{[t]}), \tag{8}$$

where $h_{ij}^{[t]}$ is the t^{th} action profile in $h_{ij}^{[\infty]}$. Formula (8) shows that $F_i(e_j)$ coincides with $V(h_{ij}^{[\infty]})$, which is the *normalized discounted value* of the infinite sequence of action profiles in the repeated path $h_{ij}^{[\infty]}$.

Note that in the framework we have presented for games with endogenous separation there is no discounting, and $F_i(e_j)$ is defined as a per-period per-player average payoff (averaged over individuals whose prevalence in t-period-old partnerships is proportional to δ^t). However, the definition of repeated path in (7) allows to establish an equivalence between $F_i(e_j)$ and the normalized discounted value $V(h_{ij}^{[\infty]})$. It follows from this equivalence that any two strategies j_1 and j_2 that generate the same repeated path against i-players obtain the same payoff against i-players, even if they have different break-up periods, i.e., even if they have different paths (as long as these paths, when repeated, generate the same sequence), i.e.:

$$h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} \implies F_{j_1}(e_i) = F_{j_2}(e_i). \tag{9}$$

3 Equilibria in games with endogenous separation: Nash and neutrally stable states

In this section we adapt standard definitions of Nash state and neutrally stable state to games with endogenous separation. For completeness, and in order to introduce the notation, we begin with the definitions for the stage game G.

3.1 Definitions for the stage game G

Here we present the main definitions and concepts for a stage game G that will be useful for the analysis of the repeated game with endogenous separation G^{Ends} derived from G.

The *best-response payoff* to action a is the best payoff that an action can obtain when playing against a, defined by

$$U^{BR}(a) \equiv \max_{a_l \in A} U(a_l, a).$$

The set of *best-response actions* to action a, $BR(a)$, is the set of actions that obtain the best-response payoff against a. If $a \in BR(a)$, i.e., if action a is a best-response to itself, we say that:

- (a, a) is a (symmetric) *Nash profile*.
- a is a *Nash action*.

The *pure minmax payoff* of G, m, is the minimum of the best-response payoffs to actions in A:

$$m \equiv \min_{a \in A} U^{BR}(a).$$

Every best-response payoff to an action is greater than or equal to m, i.e., $U^{BR}(a) \geq m \ \forall a \in A$.

A *minmax action* $\tilde{a} \in A$ is an action such that $U^{BR}(\tilde{a}) = m$. By choosing a minmax action, a player can guarantee that her opponent's payoff does not exceed m.

Let $q \in \Delta(A) \equiv \{(q_k)_{k=1}^n \in \mathbb{R}_+^n : \sum_{k=1}^n q_k = 1\}$ be a distribution over actions or *mixture* of actions. The payoff of action a against q is defined by $U_a(q) \equiv \sum_{l=1}^n U(a, a_l) q_l$. With some abuse in notation, the payoff of mixture $p \in \Delta(A)$ against $q \in \Delta(A)$ is defined by

$$U(p, q) \equiv \sum_{k=1}^n p_k \, U_{a_k}(q) = \sum_{k,l} p_k \, q_l \, U(a_k, a_l).$$

The best-response payoff against q is defined by

$$U^{BR}(q) \equiv \max_{p \in \Delta(A)} U(p, q) = \max_{a \in A} U_a(q).$$

The set of best-response actions to q, $BR(q)$, is the set of actions that obtain the best-response payoff against q.

A (symmetric) Nash equilibrium of G is a distribution $q \in \Delta(A)$ such that

$$U(q, q) = U^{BR}(q).$$

The (mixed) *minmax payoff* $\underline{m}$ of G is the mimimum of the best-response payoffs to mixtures in $\Delta(A)$:

$$\underline{m} \equiv \min_{q \in \Delta(A)} \max_{a \in A} U_a(q).$$

Every best-response payoff (to some mixture) is greater than or equal to $\underline{m}$: $U^{BR}(q) \geq \underline{m}$, i.e., independently of q, if a is a best response to q, then the payoff of a against q is at least $\underline{m}$. It follows from the definitions that $\underline{m} \leq m$.

There are several definitions of neutral stability (Maynard Smith, 1982) that are equivalent in this setting (Bomze and Weibull, 1995). Here we adopt the following one:

A distribution over actions $q \in \Delta(A)$ is *neutrally stable* if:

$$\begin{aligned} &U(q, q) \geq U(p, q) && \text{for every } p \in \Delta(A), \text{ i.e., } q \text{ is Nash, and} \\ &U(p, q) = U(q, q) \implies U(q, p) \geq U(p, p). \end{aligned}$$

Neutral stability requires that q is Nash and that it is robust to the introduction of (any combination of) alternative best responses to q, in the sense that q will not do worse than the average ($U(q,p) \geq U(p,p)$) when such alternative best responses are introduced. Neutral stability implies Lyapunov stability under the replicator dynamics (Thomas, 1985; Bomze and Weibull, 1995), and, with the previous definition of neutral stability, this result still holds if:

- The payoff functions $U_a(q)$ are defined in a different way, not necessarily linear in q, provided that they are Lipschitz continuous.
- The aggregate payoff is linear in the first argument: $U(p,q) \equiv \sum_{k=1}^{n} p_k \, U_{a_k}(q)$.
- The second condition for neutral stability applies only locally, i.e., for every p in a neighborhood of q in $\Delta(A)$.

3.2 Nash states in G^{Ends}

A strategy j is a best response to state x if, when playing against x, no other strategy (or distribution) can obtain a payoff greater than j's payoff, i.e., if and only if $F_j(x) \geq F_k(x)$ for every $k \in \Omega$. Let $BR(x)$ be the set of best-response strategies to x. A strategy distribution $y \in \mathbb{D}$ is a best response to state x if and only if $E(y,x) \geq E(z,x)$ for every $z \in \mathbb{D}$. It follows from (4) that y is a best response to x if and only if every strategy in its support $\mathbb{S}(y)$ is a best response to x.

Definition 1 (Nash equilibrium state)**.** *A state $x \in \mathbb{D}$ is Nash (short for Nash equilibrium state) if $E(x,x) \geq F_j(x)$ for every $j \in \Omega$. Equivalently, a state $x \in \mathbb{D}$ is Nash if it is a best response to itself.*

If a monomorphic state e_i is Nash, we say that strategy i is a Nash strategy. Consequently, a strategy i is Nash if and only if $F_i(e_i) \geq F_j(e_i)$ for every $j \in \Omega$.

Let us now consider some implications of being a Nash strategy. The action profiles played at a monomorphic population e_i are always symmetric, i.e. in the set $\{(a,a)\}_{a\in A}$. Consequently, the payoff $F_i(e_i)$ in a monomorphic population (see equation (5)) is a convex combination of the payoffs $\{U(a,a)\}_{a\in A}$ corresponding to the main diagonal of the payoff matrix of the stage game G. This implies that the maximum symmetric stage-game payoff $M \equiv \max_{a\in A} U(a,a)$ is an upper bound for $F_i(e_i)$.

If i is a Nash strategy, it cannot be beaten by any other strategy in its corresponding monomorphic population e_i; in particular, strategy i cannot be beaten by what we call *reap-and-leave* strategies. Reap-and-leave strategies are those which, in a partnership with i, play exactly as i up to stage $T \leq T_{ii}$, at stage T adopt a best-response action to the action chosen by i, and then break the partnership. We say that such strategies reap-and-leave i at stage T.

The fact that being Nash implies robustness against reap-and-leave strategies allows us to derive simple conditions that must be satisfied by Nash strategies and Nash states in general. The next two propositions are based on robustness against strategies that reap-and-leave i at the first stage of an i-j partnership, while the third proposition considers robustness against a strategy that reaps-and-leaves i at stage T_{ii}.

Proposition 3.1. *The first action $a^{\emptyset}$ played by a Nash strategy in G^{Ends} must satisfy*

$$U^{BR}(a^{\emptyset}) \leq M,$$

where $U^{BR}(a^{\emptyset})$ is the best-response stage payoff to action $a^{\emptyset}$ and $M = \max_{a \in A} U(a,a)$ is the maximum symmetric stage-game payoff.

We will consider the Prisoner's Dilemma and the Hawk-Dove game (also known as Snowdrift), with actions C and D (table 1). In the Prisoner's Dilemma, C stands for cooperate and D for defect; in the Hawk-Dove game, C corresponds to Dove and D to Hawk. In both cases, coordinating on C is more efficient than on D (i.e., the maximum diagonal stage payoff is $M = U_{CC} > U_{DD}$), and D is the minmax action. For the examples, we use the simpler notation $U_{a_k,a_l} \equiv U(a_k, a_l)$.

In the Prisoner's Dilemma ($U_{CD} < U_{DD} < U_{CC} < U_{DC}$), D is a dominant action and (D, D) is a Nash action profile. In the Hawk-Dove ($U_{DD} < U_{CD} < U_{CC} < U_{DC}$), the best-response to each action is the other action (this is an anti-coordination game). We will also consider the so-called 1-2-3 coordination game (table 1).

	C	D
C	3	1
D	4	2

	C	D
C	3	2
D	4	1

	1	2	3
1	1	0	0
2	0	2	0
3	0	0	3

Table 1: Left: A Prisoner's Dilemma game, with C for Cooperate and D for Defect. Middle: A Hawk-Dove game, with C for Dove and D for Hawk. Right: the 1-2-3 coordination game.

Example 3.1. *In the Prisoner's Dilemma, the maximum symmetric stage-game payoff M is U_{CC}. For action C we have $U^{BR}(C) = U_{DC} > M$, and for action D, $U^{BR}(D) = U_{DD} < M$, so action D is the only action that satisfies the condition in proposition 3.1. Consequently, every Nash strategy must begin a partnership by playing action D: no Nash strategy can be "nice" (Axelrod, 1984). This rules out strategies such as Tit for Tat, which cannot give rise to a monomorphic Nash equilibrium in a Prisoner's Dilemma with endogenous separation.*

Example 3.2. *In the Hawk-Dove, the maximum symmetric stage-game payoff M is U_{CC}. For action C (Dove) we have $U^{BR}(C) = U_{DC} > M$ and for action D (Hawk) we have $U^{BR}(D) = U_{CD} < M$, so D is the only action that satisfies the condition in proposition 3.1. Consequently, every Nash strategy must begin a partnership by playing D (Hawk).*

Example 3.3. *In 1-2-3 coordination, the maximum diagonal payoff is also the maximum possible payoff, so proposition 3.1 does not rule out any action as first action to be chosen by a Nash strategy.*

Proposition 3.2. *The minmax payoff* $\underline{m}$ *of a stage game* G *is a lower bound for the payoff at Nash states of* G^{Ends}*:*

$$x \in \mathbb{D} \text{ is Nash} \implies E(x,x) \geq \underline{m}.$$

The pure minmax payoff m *of a stage game* G *is a lower bound for the payoff* $F_i(e_i)$ *at a Nash strategy* i *of* G^{Ends}*, and* $M \equiv \max_{a \in A} U(a,a)$ *is an upper bound:*

$$i \in \Omega \text{ is Nash} \implies m \leq F_i(e_i) \leq M.$$

Example 3.4. *In the Prisoner's Dilemma, the pure minmax payoff is* $m = U_{DD}$ *and the maximum diagonal payoff is* $M = U_{CC}$*, so payoffs to Nash strategies are between the two symmetric payoffs* U_{DD} *and* U_{CC}*. For the Prisoner's Dilemma on table 1, payoffs to Nash strategies are between 2 and 3.*

Example 3.5. *In the Hawk-Dove, the pure minmax payoff is* $m = U_{CD}$ *and the maximum diagonal payoff is* $M = U_{CC}$*, so payoffs to Nash strategies are between the two payoffs for a C-player (Dove):* U_{CD} *and* U_{CC}*. For the Hawk-Dove on table 1, payoffs to Nash strategies are between* 2 *and* 3*.*

Example 3.6. *In 1-2-3 coordination, the pure minmax payoff is* $m = U_{11}$ *and the maximum diagonal payoff is* $M = U_{33}$*, so payoffs to Nash strategies are between* U_{11} *and* U_{33}*. For the 1-2-3 coordination game on table 1, payoffs to Nash strategies are between* 1 *and* 3*.*

Proposition 3.3. *If* i *is a Nash strategy with finite* T_{ii}*, then the action profile at the break-up stage* T_{ii} *of an* i*-*i *partnership is a Nash profile of the stage game* G*.*

Example 3.7. *In a Prisoner's Dilemma with endogenous separation, the action profile at the break-up stage of a Nash strategy with finite* T_{ii} *has to be (D,D).*

Example 3.8. *In a Hawk-Dove game, neither (C,C) nor (D,D) are Nash profiles, so in a Hawk-Dove game with endogenous separation there is no Nash strategy* i *with finite* T_{ii}*.*

For the Prisoner's Dilemma, Proposition 3.4 below strengthens the previous result.

Proposition 3.4. *In the Prisoner's Dilemma with endogenous separation, Nash strategies with finite* T_{ii} *never play* C *in the equilibrium path.*

Proposition 3.4 follows from considering that, in the Prisoner's Dilemma with endogenous separation, if a strategy i with finite T_{ii} ever plays the action profile (C,C) in an i-i partnership, then there is a stage T_l in $[1, T_{ii}]$ at which (C,C) is played for the last time, and a strategy j that reaps-and-leaves i at stage T_l beats i (in the sense $F_j(e_i) > F_i(e_i)$), so i cannot be Nash. Proposition 3.4 can be extended to games G with only one symmetric Nash action profile which is the least efficient of the symmetric action profiles.

Proposition 3.5. *Let (a^N, a^N) be a Nash profile of G.*

- *Every strategy i that always chooses action a^N before breaking a partnership is a Nash strategy of G^{Ends}.*
- *Any mixture of such strategies is a Nash state of G^{Ends}.*

Example 3.9. *In a Prisoner's Dilemma with endogenous separation, any strategy i that for every history of length between 0 and T_{ii} (for some $T_{ii} > 0$) plays D, and breaks every partnership that gets to stage T_{ii}, is a Nash strategy (i.e., e_i is a monomorphic Nash state). Any mixture of such strategies is a Nash (polymorphic) state.*

Example 3.10. *In a Hawk-Dove game, neither (C,C) nor (D,D) are Nash profiles, so we cannot use proposition 3.5 to construct Nash strategies for the game with endogenous separation.*

Example 3.11. *In 1-2-3 coordination, we can use proposition 3.5 to construct Nash strategies and Nash states for the game with endogenous separation from either of the three Nash action profiles of the stage game.*

One can derive general existence theorems for Nash equilibria in games with endogenous separation for sufficiently large values of δ (folk theorems). We will do so in the next section using the (stronger) concept of neutral stability.

3.3 Neutrally stable states in G^{Ends}

In this section we define neutrally stable strategy and neutrally stable state in games with endogenous separation. Our definitions are direct adaptations of a standard definition of neutral stability (Banerjee and Weibull, 2000): a state is neutrally stable if it is (i) a best response to itself, and also (ii) a weakly-better response to all its best-response states (than such states are to themselves).

Definition 2 (Neutrally stable strategy)**.** *A strategy $i \in \Omega$ is neutrally stable if*

$$F_i(e_i) \geq F_j(e_i) \qquad \text{for every } j \in \Omega, \text{ i.e., } i \text{ is Nash, and}$$
$$F_i(y) \geq E(y, y) \qquad \text{for every } y \in \mathbb{D} \text{ such that } E(y, e_i) = F_i(e_i).$$

Note that, given that i is Nash, condition $E(y, e_i) = F_i(e_i)$ means that y is a mixture of best-response strategies to e_i. Also, note that the definition of neutral stability requires strategy i to satisfy $F_i(y) \geq E(y, y)$ whenever y is a mixture of alternative best-response strategies to e_i. This robustness to every possible *mixture* of best response strategies is stronger than robustness against all best response strategies considered individually, as defined by the following condition:

$$F_i(e_j) \geq F_j(e_j) \text{ for every } j \in \Omega \text{ that is a best-response to } e_i.$$

This latter condition is necessary but not sufficient for neutral stability. The reason is that, if j_1 and j_2 are two best-response strategies to e_i, and y is a mixture of j_1 and

j_2, then $F(y, y)$ depends not only on the payoffs $F_{j_1}(e_{j_1})$ and $F_{j_2}(e_{j_2})$ of each best-response strategy against itself, but also on the payoffs $F_{j_1}(e_{j_2})$ and $F_{j_2}(e_{j_1})$ for the crossed interactions.

Definition 3 (Neutrally stable state). *A state $x \in \mathbb{D}$ is neutrally stable if*

$$\begin{aligned} E(x, x) &\geq E(y, x) && \text{for every } y \in \mathbb{D}, \text{ i.e., } x \text{ is Nash, and} \\ E(x, y) &\geq E(y, y) && \text{for every } y \in \mathbb{D} \text{ such that } E(y, x) = E(x, x). \end{aligned}$$

It follows from the corresponding definitions that a monomorphic state e_i is neutrally stable if and only if strategy i is neutrally stable.

Considering a finite set of strategies, we can define the replicator dynamics for games with endogenous separation as a direct adaptation of the standard replicator dynamics (Taylor and Jonker, 1978).

Definition 4 (Replicator dynamics). *Consider a game with endogenous separation and any finite set of strategies $S \subset \Omega$. Let the replicator dynamics in S be the set of differential equations*

$$\dot{x}_i = x_i[F_i(x) - E(x, x)] \tag{10}$$

for $i \in S$.

By numbering the s strategies in S, we can associate every strategy distribution with support contained in S with a point in the standard simplex $\Delta(S) \subset \mathbb{R}^s$.[6]

Proposition 3.6. *Let $x \in \mathbb{D}$ be neutrally stable, and let S be any finite (numbered) superset of its support. Let $\hat{x} \in \Delta(S)$ be the point that represents x in $\mathbb{R}^s$. Then $\hat{x}$ is a Lyapunov stable rest point of the (standard) replicator dynamics in $\Delta(S)$.*

Proposition 3.6 shows that, if x is neutrally stable according to definition 3, then its associated point $\hat{x}$ representing x in $\Delta(S)$ is a Lyapunov stable rest point in the replicator dynamics, considering any finite set of strategies S that includes:

- the support of x (*incumbents*), and
- any other set of strategies (*potential invaders*), whatever those potential invaders may be.

Proposition 3.7 below shows a strong limitation for the stability of strategies with finite break-up period: if none of the symmetric action profiles of G that obtain the maximum symmetric payoff $M = \max_{a \in A} U(a, a)$ are Nash, then no strategy i with finite break-up period T_{ii} can be neutrally stable.

Proposition 3.7. *For stage game G, let $A_M \equiv \{(a_k, a_k) : U(a_k, a_k) = M\}$ be its set of symmetric action profiles with payoff $M = \max_{a \in A} U(a, a)$.*

[6] The s strategies in S can be numbered following any order $i_1, i_2, ..., i_{s-1}, i_s$, and then we can define $\hat{x}_k = x_{i_k}$ for $k \in \{1, ..., s\}$.

- *If A_M does not contain any Nash profile of G, then no strategy i with finite T_{ii} is neutrally stable in G^{Ends}.*
- *Otherwise, any neutrally stable strategy i with finite T_{ii} obtains the payoff $F_i(e_i) = M$, and the action profiles played in its equilibrium e_i are Nash profiles contained in A_M.*

Example 3.12. *In the Prisoner's Dilemma and in the Hawk-Dove game, the symmetric action profile with maximum stage payoff is (C, C), which is not a Nash profile. Consequently, in the Prisoner's Dilemma and in the Hawk-Dove game with endogenous separation, no strategy with finite break-up period T_{ii} is neutrally stable.*

Example 3.13. *In 1-2-3 coordination, the symmetric action profile with maximum payoff is (a_3, a_3), which is a Nash profile. Consequently, any neutrally stable strategy with finite T_{ii} must always play (a_3, a_3) in the equilibrium path.*

4 Stronger equilibrium concepts. Path-protecting strategies

In this section we discuss equilibrium concepts stronger than neutral stability. We first consider *evolutionarily stable strategies*, which cannot exist in games with endogenous separation. We then define two weaker equilibrium concepts, namely *path-protecting* and *weakly path-protecting* strategies, which imply neutral stability and whose existence is guaranteed in many games with endogenous separation, for sufficiently large values of δ.

4.1 Evolutionarily stable strategies

We start by considering the standard static equilibrium concept of evolutionary stability (Maynard Smith and Price, 1973).

Definition 5 (Evolutionarily stable strategy)**.** *A strategy $i \in \Omega$ is evolutionarily stable if*

$$\begin{aligned} &F_i(e_i) \geq F_j(e_i) && \text{for every } j \in \Omega, \text{ i.e., } i \text{ is Nash, and} \\ &F_i(y) > E(y, y) && \text{for every } y \neq e_i \in \mathbb{D} \text{ such that } E(y, e_i) = F_i(e_i). \end{aligned}$$

Evolutionary stability for a Nash strategy i requires that there are no alternative best-response strategies j with $F_i(e_j) = F_j(e_j)$. The concept of *path-equivalent strategy*, defined below, will be useful to show that, in games with endogenous separation, there are no evolutionarily stable strategies. The argument extends easily to polymorphic states with finite support, and is basically the same argument used to show that there are no evolutionarily stable strategies in standard repeated games (Selten and Hammerstein, 1984).

Definition 6 (Path-equivalent strategy)**.** *Strategy j is path-equivalent to strategy i if*

$$a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}.$$

Considering that the action profiles in $a_{ii}^{[1,T_{ii}]}$ are symmetric, it follows that if j is path-equivalent to i, then $a_{ii}^{[1,T_{ii}]} = a_{ji}^{[1,T_{ji}]} = a_{ij}^{[1,T_{ij}]} = a_{jj}^{[1,T_{jj}]}$ and, consequently, $F_i(e_i) = F_j(e_i) = F_i(e_j) = F_j(e_j)$. If i is Nash and j is path-equivalent to i, then j is an alternative best-response to i (i.e., $F_j(e_i) = F_i(e_i)$) that satisfies $F_i(e_j) = F_j(e_j)$. By modifying the choices made by strategy i after histories $a^{[0,t]}$ that do not belong to the set of histories $\{a_{ii}^{[0,t]}\}_{t\in[0,T_{ii}]}$ generated by an i-i partnership, one can create (an infinite number of) strategies that are path-equivalent to strategy i. This proves that no strategy is evolutionarily stable in a game with endogenous separation.

For completeness, in appendix B we discuss another equilibrium concept stronger than neutral stability: robustness against indirect invasions (van Veelen, 2012). We show that in many games with endogenous separation, such as those whose stage game is the Prisoner's Dilemma or the Hawk-Dove game, no strategy can be robust against indirect invasions.

4.2 Path-protecting strategies

In this section we define *path-protecting* and *weakly path-protecting* strategies. Both concepts imply neutral stability. Their existence will be discussed later.

Definition 7 (Path-protecting strategy)**.** *A strategy $i \in \Omega$ is path-protecting if:*

$$a_{jj}^{[1,T_{jj}]} \neq a_{ii}^{[1,T_{ii}]} \implies F_j(e_i) < F_i(e_i).$$

In words, a strategy i is path-protecting if, when playing against i, only those strategies that are path-equivalent to i (those with $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}$) obtain the same payoff as i, while every strategy j that is not path-equivalent to i obtains a strictly lower payoff.

Note that a necessary condition for a strategy i to be path-protecting is that $T_{ii} = \infty$. The reason is that, if T_{ii} is finite, any strategy j with $T_{jj} > T_{ii}$ whose path of play up to stage T_{ii} coincides with that of i (i.e., $a_{jj}^{[1,T_{ii}]} = a_{ii}^{[1,T_{ii}]}$) satisfies $F_j(e_i) = F_i(e_i)$.

Considering that $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,\infty]}$ if and only if $a_{ij}^{[1,T_{ij}]} = a_{ii}^{[1,\infty]}$, an equivalent definition for a path-protecting strategy i is: a strategy i is path-protecting if $T_{ii} = \infty$ and

$$a_{ji}^{[1,T_{ij}]} \neq a_{ii}^{[1,\infty]} \implies F_j(e_i) < F_i(e_i).$$

This alternative definition shows that a path-protecting strategy i "protects" the equilibrium path against strategies that, when playing with i, deviate at some point from i's choice, either by choosing a different action or by breaking the partnership.

We now define a concept weaker than path-protecting strategy, which will turn out to be sufficient to guarantee neutral stability, namely *weakly path-protecting strategy.* Before doing so, for convenience, let us recall that $h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]})^{\infty}$ is the infinite sequence of action profiles generated by strategy i in a population of j-players with no exogenous separation (7), and $F_i(e_j)$ coincides with $V(h_{ij}^{[\infty]})$, the normalized discounted value of (the action profiles in) $h_{ij}^{[\infty]}$.

Definition 8 (Weakly path-protecting strategy)**.** *A strategy $i \in \Omega$ with $T_{ii} = \infty$ is weakly path-protecting if:*

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_j(e_i) < F_i(e_i).$$

In words, a strategy i is weakly path-protecting if

- it never breaks a partnership with a partner who takes the same actions as i does, and
- if the repeated path $h_{ji}^{[\infty]}$ that strategy j generates in a population of i-players is different from the path that i generates, then j obtains a strictly lower payoff (in a population of i-players) than i.

Note that any strategy j that at some stage of an i-j partnership adopts a different action from the action adopted by i generates a different repeated path $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$. Strategies j that, before breaking an i-j partnership at a finite stage T_{ij}, do not adopt different actions from i's, may still generate the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$, but only if $h_{ii}^{[\infty]}$ is an infinite repetition of the finite sequence of T_{ij} (symmetric) action profiles $a_{ii}^{[1,T_{ij}]} = a_{ji}^{[1,T_{ij}]}$.

For any strategy i with a path $a_{ii}^{[1,\infty]}$ that is not an infinite repetition $(a^{[1,T]})^{\infty}$ of some finite sequence $a^{[1,T]}$ of action profiles, being weakly path-protecting is equivalent to being path-protecting, because, in that case, the only way a strategy i can protect the repeated path $h_{ii}^{[\infty]}$ is by protecting the path $a_{ii}^{[1,\infty]}$. By contrast, strategies i with a path $a_{ii}^{[1,\infty]}$ that is an infinite repetition of some finite sequence may be weakly path-protecting, but cannot be path-protecting.

Considering equation (9), it follows from definition 7 that if strategy i is weakly path-protecting, then:

- Strategy i is Nash, because strategies with the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$ obtain the same payoff $F_j(e_i) = F_i(e_i)$ and strategies with different repeated path obtain a lower payoff $F_j(e_i) < F_i(e_i)$, so $F_j(e_i) \leq F_i(e_i)$ for every j.
- Every best-response strategy j to e_i must generate the same (symmetric) repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$. This implies that, if j is a best-response to e_i, then $F_i(e_j) = F_j(e_i) = F_i(e_i)$. It also implies that if y is a mixture of best-response strategies to e_i, then $F_i(y) = F_i(e_i)$.

Our next result states that (weakly) path-protecting strategies are neutrally stable. Its proof shows that, if strategy i is weakly path-protecting, then any mixture y of best-response strategies to e_i must satisfy $E(y, y) = F_i(y)$. The reason is that every repeated path $h_{j_1 j_2}^{[\infty]}$ generated between any two best-response strategies (j_1 and j_2) to e_i must also be equal to $h_{ii}^{[\infty]}$, so if y is a mixture of best-response strategies to e_i, then $E(y, y) = F_i(e_i) = F_i(y)$.

Proposition 4.1. *(Weakly) path-protecting strategies are neutrally stable.*

Weakly path-protecting strategies can be easily found if the stage game has some strict Nash profile, as our next result shows.

Proposition 4.2. *If $(\hat{a}, \hat{a})$ is a strict Nash profile of a stage game G, then any strategy of G^{Ends} that:*

- *chooses action $\hat{a}$ whenever it does not choose to break a partnership, and*
- *does not break a partnership while profile $(\hat{a}, \hat{a})$ is played*

is weakly path-protecting (and, consequently, neutrally stable).

Example 4.1. *In the Prisoner's Dilemma, (D, D) is a strict Nash profile. Consequently, any strategy that never plays C and never breaks up after a history of mutual defections is weakly path-protecting. For instance, the strategy "always play D and never leave", that maps every history to D, is weakly path-protecting and, consequently, neutrally stable.*

Much more generally than the case in which G has some strict Nash profile, Proposition 4.3 below shows that, for large enough δ, every stage game G with $M > m$ admits path-protecting strategies. Proposition 4.3 leads to a *folk theorem* for neutral stability which basically says that, for large enough δ, any payoff between m and M can be obtained as the equilibrium payoff of some neutrally stable strategy.

Before stating proposition 4.3, let us define the average stage-payoff for a finite sequence of action profiles. Considering a sequence $\Phi = (\Phi^{[t]})_{t=1}^{T}$ of T action profiles, where each $\Phi^{[t]} \in A^2$ is an action profile, let the average stage-payoff of sequence Φ be

$$\bar{U}_\Phi \equiv \frac{\sum_{t=1}^{T} U(\Phi^{[t]})}{T}.$$

The average stage payoff is specially relevant for large δ and for paths that end up repeating some sequence Φ of action profiles, because the normalized payoff of any infinite path $([...], \Phi, \Phi, \Phi, ...)$ which, after a finite number of periods, eventually repeats the finite sequence of outcomes Φ forever, converges to the average stage-payoff $\bar{U}_\Phi$ as δ goes to 1.

Proposition 4.3. *Let Φ be a finite sequence of symmetric action profiles with average stage payoff $\bar{U}_\Phi$ strictly greater than the pure minmax payoff. For large enough $\delta < 1$, there are path-protecting strategies whose equilibrium path, after a finite transient phase, is an infinite repetition of the sequence Φ, and whose equilibrium payoff converges to $\bar{U}_\Phi$ as $\delta \to 1$.*

Considering that $\bar{U}_\Phi$ can approximate any real payoff between m and M as much as desired, Proposition 4.3 has as a corollary the following folk theorem for neutrally stable strategies.

Corollary 4.4. *In a game with endogenous separation, for large enough values of the continuation probability δ, any payoff between the pure minmax payoff m and the maximum symmetric payoff M of the stage game can be obtained, or approximated as much as desired, as the equilibrium payoff of some neutrally stable strategy.*

The proof of proposition 4.3 is included in appendix D, but here we provide a sketch. The proof is constructive and considers a strategy i such that:

- It never breaks a partnership with a partner who takes the same actions as i does (i.e., $T_{ii} = \infty$).
- As soon as strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership. Because of this condition, we know that an i-j partnership will not survive if j chooses a different action from the action chosen by i. Naturally, it will not survive either if j chooses to break the partnership. The only way in which an i-j partnership can survive indefinitely is if j chooses the same initial action as i does and, for every history $a_{ii}^{[0,t]}$ corresponding to an i-i partnership, j chooses the same action as i does.
- The path $a_{ii}^{[1,\infty]}$ is made up by three phases, each one associated to one finite sequence of symmetric action profiles (Φ_m, Φ_f and Φ_p), with

  $$a_{ii}^{[1,\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^\infty),$$

  where Φ_m is a repetition of a minmax action profile, Φ_f is arbitrary (but finite), Φ_p (which corresponds to the infinitely repeated pattern Φ in proposition 4.3) has an average stage payoff greater than the pure minmax payoff m of the stage game, and $(\Phi_p)^\infty$ represents an infinite sequence of action profiles made up by repeating the sequence Φ_p infinitely.
- The first phase in $a_{ii}^{[1,\infty]}$ is a T_m-period-long phase during which a minmax action profile $(\tilde{a}, \tilde{a})$ is played, producing the sequence

  $$\Phi_m = a_{ii}^{[1,T_m]} = ((\tilde{a}, \tilde{a}), (\tilde{a}, \tilde{a}), ..., (\tilde{a}, \tilde{a})).$$

  During this minmax or *deviation-deterring* phase, the stage payoff is $U(\tilde{a}, \tilde{a}) \leq m$ and any strategy j that deviates in choice during this phase obtains a payoff $F_j(e_i) \leq m$.
- The second phase in the path $a_{ii}^{[1,\infty]}$ is an arbitrary finite sequence of $T_f \geq 0$ (symmetric) action profiles.
- The last phase, or *pattern-playing* phase, in $a_{ii}^{[1,\infty]}$ is an infinite repetition of a finite sequence (pattern) Φ_p of T_p symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

The proof of proposition 4.3 combines three intermediate results to create path-protecting strategies. These strategies are initially built to be weakly path-protecting, and then fine-tuned so the path when they play against themselves is not an infinite repetition of any finite sequence of action profiles, so they are also path-protecting.

- The first result shows that, in order to prove that the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_j(e_i) < F_i(e_i)$ holds for every strategy j, it is enough to prove that it holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation before and up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful.
- The second result states that, for any given Φ_f and Φ_p (with $\bar{U}_{\Phi_p} > m$), the deviation-deterring phase can be chosen to be long enough to guarantee that, for sufficiently large δ, deviations in $h_{ji}^{[\infty]}$ from $h_{ii}^{[\infty]}$ at or before $t = T_m + T_f + T_p$ lead to payoffs $F_j(e_i)$ close to or below m.
- The third result states that, for sufficiently large δ, the payoff $F_i(e_i)$ is close to $\bar{U}_{\Phi_p} > m$.

Combining the three results shows that, given Φ_f and Φ_p, there is a length of the deviation-deterring phase T_m such that, for large enough δ, $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$ implies $F_j(e_i) < F_i(e_i)$, so strategy i is weakly path protecting. Finally, by choosing Φ_p so that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, we make sure that strategy i is also path-protecting.

Example 4.2. *In a Prisoner's Dilemma or in a Hawk-dove game, the minmax profile is DD,*[7] *so:*

- *For the deviation-deterring or minmax phase, Φ_m is a T_m-long series of DD action profiles.*
- *For the pattern-playing phase, the infinitely repeated finite pattern Φ_p can be any finite sequence of DD and CC action profiles with at least one CC in the sequence, which guarantees an average stage payoff $\bar{U}_{\Phi_p} > m = U_{DD}$.*

For instance, choosing $T_m = 3$, $\Phi_f = (CC, DD)$ and $\Phi_p = (CC)$, we obtain a strategy i with path $h_{ii}^{\infty} = (DD, DD, DD \,|\, CC, DD \,|\, (CC)^{\infty})$. For the stage payoffs shown on table 1 for the Prisoner's Dilemma, the sequence of payoffs corresponding to h_{ii}^{∞} is $(2, 2, 2, 3, 2, (3)^{\infty})$, where $()^{\infty}$ represents an infinite repetition of the payoffs in brackets, so

$$F_i(e_i) = (1-\delta)(2 + 2\delta + 2\delta^2 + 3\delta^3 + 2\delta^4 + 3\frac{\delta^5}{1-\delta}) > 2.$$

The pattern $\Phi_p = (CC)$ begins to be repeated after period 6. Strategies j with $T_{ji} < 3$ obtain a payoff $F_j(e_i)$ of at most the minmax payoff $2 < F_i(e_i)$. For $T_{ji} = 4$ the payoff

[7] For compactness, here we represent action profiles (D, D) as DD.

$F_j(e_i)$ is bounded by that of the series $(2,2,2,4)^\infty$, and for $5 \le T_{ji} \le 6$ the payoff is bounded by that of the series $(2,2,2,3,2,4)^\infty$. For $\delta > 0.71$, $F_i(e_i)$ is greater than the payoffs corresponding to both series, so i is path-protecting.

Example 4.3. *In 1-2-3 coordination, the minmax profile is (a_1, a_1), so*

- *Φ_m is a T_m-long series of (a_1, a_1) action profiles.*
- *Φ_p can be any finite sequence of symmetric action profiles where at least one action profile is not (a_1, a_1), which guarantees $\bar{U}_{\Phi_p} > m = U(a_1, a_1)$.*

5 Polymorphic neutrally stable states

Let us now consider polymorphic neutrally stable states, in which players in a population use different strategies (beyond those already considered in proposition 3.5).

One can start by looking for polymorphic states made up by combinations of (weakly) path-protecting strategies, but our next result shows that, if two weakly path-protecting strategies i and j have different paths $a_{ii}^{[1,\infty]} \neq a_{jj}^{[1,\infty]}$, then they cannot both be in the support of a neutrally stable state. Consequently, there are no neutrally stable states with more than one (weakly) path-protecting strategy, unless the different strategies are actually generating the same repeated path.

Proposition 5.1. *If a neutrally stable state x has some (weakly) path-protecting strategy i in its support then all the repeated paths in x are equal to $h_{ii}^{[\infty]} = a_{ii}^{[1,\infty]}$.*

Proposition 5.1 shows that mixtures of path-protecting strategies with different paths do not satisfy definition 3 of neutral stability.

We conclude with a series of definitions and a proposition that allow us to extend some of the previous concepts to polymorphic states.

Definition 9 (Path-equivalent strategy in a set)**.** *Let S be a finite set of strategies satisfying $T_{ij} = \infty$ for every $i, j \in S$. We say that strategy k is path-equivalent in S to strategy $i \in S$ if, for every $j \in S$,*

$$T_{kj} = \infty \text{ and } a_{kj}^{[1,\infty]} = a_{ij}^{[1,\infty]}.$$

The idea here is that, with each of the strategies in S, strategy k behaves exactly as strategy i does, and there is no difference also between $a_{ii}^{[1,\infty]}$ and $a_{kk}^{[1,\infty]}$.

Definition 10 (Path-protecting state)**.** *A population state x with finite support $\mathbb{S}(x)$ is path-protecting if:*

- *$T_{ij} = \infty$ for every $i, j \in \mathbb{S}(x)$, and*
- *If strategy j is not path-equivalent in $\mathbb{S}(x)$ to some strategy $i \in \mathbb{S}(x)$, then $F_j(x) < E(x, x)$.*

It follows from the definition that path-protecting states are Nash.

Definition 11 (Internally neutrally stable state)**.** *A state x is internally neutrally stable if $F_i(x) = E(x,x)$ for every $i \in \mathbb{S}(x)$ and $F(x,y) \geq F(y,y)$ for every y with support contained in $\mathbb{S}(x)$.*

This condition only considers strategies in the support of state x, and it is clearly a necessary condition for neutral stability, which considers the whole strategy space.

Proposition 5.2. *If a state is path-protecting and internally neutrally stable, then it is neutrally stable.*

In appendix C we present an example of a path-protecting and internally neutrally stable equilibrium, for a Prisoner's Dilemma with endogenous separation.

6 Conclusions

In the standard approach to repeated games, partners are tied to each other and do not have a say on whether they wish to stay together or whether they prefer to leave the partnership and form a new one. For many real-life situations, the field of games with endogenous separation constitutes a natural and realistic extension of the standard approach to repeated games.

Following some pioneers (most notably, Carmichael and MacLeod (1997)), a major step forward to study games with endogenous separation was taken by Fujiwara-Greve and Okuno-Fujiwara (2009).[8] This seminal paper, while focused on the Prisoner's Dilemma, provided the first fundamental framework for the study of games with endogenous separation taking into account the whole strategy space.[9] Fujiwara-Greve and Okuno-Fujiwara's (2009) framework is based on the strategy distribution in the *pool of singles*. In contrast, in this paper we develop an approach based on the distribution of strategies in the population, or population state.[10] This approach allows us to establish clear links and differences between games with endogenous separation and standard repeated games, including the definition of appropriate payoff functions for (a group of) potential invaders, the adaptation of static equilibrium concepts such as neutral stability, and the adaptation of standard dynamics such as the replicator dynamics.

In this paper, we have also introduced the notion of *path-protecting* strategy (an equilibrium concept stronger than neutral stability), and provided an existence result for path-protecting strategies in games with endogenous separation: in general, for sufficiently large continuation probability, there is a large variety of path-protecting neutrally

[8]It is also worth noting the work of Vesely and Yang (2010), which constitutes an approach based on behavioral strategies.

[9]This framework has been used and extended in subsequent papers such as Fujiwara-Greve et al. (2012, 2015).

[10]Izquierdo et al. (2021) can be considered an intermediate approach that combines the pool and population states, which allows them to prove some relevant properties of the payoff functions.

stable strategies. The concept of path-protecting strategy generalizes the idea of *trust-building* strategies that appear in previous related works for the Prisoner's Dilemma and some of its variations (Datta, 1996; Ghosh and Ray, 1996; Carmichael and MacLeod, 1997; Kranton, 1996; Fujiwara-Greve and Okuno-Fujiwara, 2009). We have also extended the concept of path-protecting strategy from strategies (monomorphic states) to mixtures of strategies in a population (polymorphic states).

Extensions of the framework of games with endogenous separation to multiplayer asymmetric games or multi-population games present additional challenges and remain an open field of research.

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A Other approaches to neutral stability in games with endogenous separation

Here we present previous definitions of neutral stability for games with endogenous separation. Several of these definitions are based on a condition for (neutral) stability from Taylor and Jonker (1978), which can be adapted as follows (Bomze and Weibull, 1995):

Definition 12. *Considering a finite set of strategies S, a state $x \in \Delta(S)$ is neutrally stable NSS_{TJ} in $\Delta(S)$ if for every $y \in \Delta(S)$ there is some $\bar{\epsilon}_y \in (0,1)$ such that*

$$F(x, \epsilon\, y + (1-\epsilon)\, x) \geq F(y, \epsilon\, y + (1-\epsilon)\, x)$$

for all $\epsilon \in (0, \bar{\epsilon}_y)$.

Definition 13. *Carmichael and MacLeod (1997). A Nash equilibrium population state x is a neutrally stable state NSS_1 if for every $y \in \mathbb{D}$ there exists an $\bar{\epsilon}_y \in (0,1)$ such that for every $\epsilon \in (0, \bar{\epsilon}_y)$,*

$$F_i(\,(1-\epsilon)x + \epsilon y) \geq F_j(\,(1-\epsilon)x + \epsilon y)$$

for all $i \in supp(x)$ and $j \in supp(y)$.

Definition 14. *Fujiwara-Greve and Okuno-Fujiwara (2009). A distribution in the matching pool p is a neutrally stable pool distribution NSS_2 if for every $j \in \Omega$ there exists an $\bar{\epsilon}_j \in (0,1)$ such that for every $\epsilon \in (0, \bar{\epsilon}_j)$ and every $i \in supp(x)$,*

$$F_i^p(\,(1-\epsilon)\,p+\epsilon\,e_j) \geq F_j^p(\,(1-\epsilon)\;p+\epsilon\,e_j)$$

It is easy to see that, even when considering a restricted set of strategies S, the conditions for NSS_1 and NSS_2 are not equivalent to NSS_{TJ}. Izquierdo et al. (2021, Appendix C) present simple examples of states that are NSS_{TJ} but are not NSS_1 or NSS_2, as well as examples of states that are NSS_2 but not NSS_{TJ} and which are not stable in the replicator dynamics.

Considering behavioral strategies, Vesely and Yang (2010) provide a definition of neutral stability for games with endogenous separation that is closer to the condition for NSS_{TJ}. However, an important point to consider is that, if the payoff functions $F_i(x)$ are not linear (and this is generically the case in games with endogenous separation), being NSS_{TJ} does not guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$ (Bomze and Weibull (1995)). In contrast, the condition that we use for neutral stability does guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$.

Izquierdo et al. (2021) provide a definition of neutral stability that looks rather involved because it uses the population and pool distributions simultaneously, but which can be shown to be equivalent to the following simplification:

Definition 15. *Izquierdo et al. (2021). A population state x is a neutrally stable state of a game with endogenous separation (NSS_3) if it is a Nash equilibrium and for any finite set of strategies S with $\mathbb{S}(p) \subseteq S$ there is a neighborhood O_S of x in $\Delta(S)$ such that $F(x,y) \geq F(y,y)$ for every $y \in O_S$ with $F(y,x) = F(x,x)$.*

It is easy to see that our condition for neutral stability (definition 3) involves satisfaction of the condition for NNS_3.

When comparing our results with those in Carmichael and MacLeod (1997), Fujiwara-Greve and Okuno-Fujiwara (2009) or Izquierdo et al. (2021), the reader should keep in mind the different definitions of neutral stability used in each of those papers.

B Strategies robust against indirect invasions

Here we consider the equilibrium condition of robustness against indirect invasions or RAII (van Veelen, 2012) for games with endogenous separation. It can be argued that any reasonable extension of this concept to games with endogenous separation would require at least neutral stability and that every weakly path-equivalent strategy is also neutrally stable, where j is said to be weakly path-equivalent to i if $h_{jj}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$ (the second equality is implied by the first), which implies that any mixture y of strategies i and j satisfy $E(y,e_i) = F_i(e_i) = F_i(y) = E(y,y)$. With these minimum requirements, our results below show that, in many cases of interest, there are no RAII strategies in games with endogenous separation. We first show that being RAII requires playing Nash action profiles of the stage game and, in most cases of interest, it requires $T_{ii} = \infty$ and a sufficiently low value of δ. For (sufficiently) large values of δ, and unless the maximum payoff of the stage game is attained at a symmetric Nash action profile, no strategy is robust against indirect invasions. The reason is that every strategy i has a path-equivalent strategy j_1 that would let a potential invader j_2 who deviates in action (from i or j_1) at the first stage of an j_2-j_1 partnership obtain the maximum stage game payoff afterwards, in an infinite path $a_{j_2 j_1}^{[1,\infty]}$. The payoff $F_{j_2}(e_{j_1})$ to such a strategy j_2 converges to the maximum stage game payoff as $\delta \to 1$.

Proposition B.1. *A strategy $i \in \Omega$ can be robust against indirect invasions only if the action profiles played in the i-i equilibrium path are Nash profiles of the stage game.*

It follows from proposition 3.7 that, unless the maximum symmetric payoff of the stage game corresponds to a Nash profile, $T_{ii} = \infty$ is also a necessary condition for a strategy to be RAII, as it is a necessary condition for neutral stability.

Proof of proposition B.1. Suppose that the action profile $(a_i^{[t]}, a_i^{[t]}) = (a, a)$ is not Nash. Consider two strategies j and k such that:

- Strategy j is path-equivalent to i, so $F_j(e_j) = F_i(e_i)$.
- Strategy k behaves with j (or with i) like j up to stage t (i.e., $a_{kj}^{[1,t-1]} = a_{jj}^{[1,t-1]} = a_{ii}^{[1,t-1]}$ if $t > 1$) and deviates at t by playing a best response action to action a, obtaining at that stage a greater payoff than what j obtains in a j-j partnership.
- From stage t, strategies j and k do not break up and play the action profile that provides k the maximum possible payoff of the stage game.

Then $F_k(e_j) > F_j(e_j)$, so strategy j is not Nash. □

Proposition B.2. *For stage games with a single Nash action profile which does not obtain the maximum symmetric payoff, such as the Prisoner's Dilemma, no strategy in the game with endogenous separation is RAII.*

Proof of proposition B.2. The only possible candidates to be RAII are strategies with $T_{ii} = \infty$ that always play the Nash action profile at the equilibrium. But any such strategy i has a weakly path-equivalent strategy j with finite T_{jj} that always plays the Nash action profile in $a_{jj}^{[1,T_{jj}]}$, and which, by proposition 3.7, is not neutrally stable. □

Example B.1. *For the Prisoner's Dilemma, the only Nash action profile is (D, D) and it does not obtain the maximum symmetric payoff U_{CC}, so there are no RAII strategies in the game with endogenous separation.*

Example B.2. *For the Hawk-Dove game, no symmetric action profile is Nash, so there are no RAII strategies in the game with endogenous separation.*

C Example of a bimorphic path-protecting equilibrium

Consider a Prisoner's Dilemma game with the payoffs shown in table 2. For the game with endogenous separation, let strategy 1 and strategy 2 be two strategies that generate the paths $a_{ij}^{[1,\infty]}$ shown in table 2, with the corresponding payoffs $F_i(e_j)$ shown in table 3. Strategy 1 is such that, if an opposing strategy j generates in a j-1 partnership a history that is not coherent with either $a_{11}^{[1,\infty]}$ or $a_{21}^{[1,\infty]}$, strategy 1 breaks up the partnership. In the same way, strategy 2 breaks any j-2 partnership as soon as the history deviates from both $a_{12}^{[1,\infty]}$ and $a_{22}^{[1,\infty]}$.

	C	D
C	3	-1
D	5	0

	1	2
1	$(DD)^{T_1}\,(CC)^{\infty}$	$(DD)^{T_2}\,DC\,(CC)^{\infty}$
2	$(DD)^{T_2}\,CD\,(CC)^{\infty}$	$(DD)^{T_2}(CC)^{\infty}$

Table 2: Left: Stage game payoffs for a Prisoner's Dilemma, with C for Cooperate and D for Defect. Right: Paths $a_{ij}^{[1,\infty]}$ that strategy 1 and strategy 2 generate together, with i for the row strategy and j for the column. It is assumed that $T_1 > T_2$

	1	2
1	$\delta^{T_1}\,3$	$\delta^{T_2}[5\,(1-\delta)+3\,\delta]$
2	$\delta^{T_2}[(-1)(1-\delta)+3\,\delta]$	$\delta^{T_2}\,3$

Table 3: Payoffs $F_i(e_j)$ corresponding to the paths shown in table 2.

Let us take $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$, leading to the $F_i(e_j)$ payoffs shown in table 4.

	1	2
1	1.59	2.10
2	1.71	1.97

Table 4: Payoffs $F_i(e_j)$ corresponding to the paths shown in table 2, for $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$.

At a population state made up by strategies 1 and 2 in proportions x_1 and x_2, considering that all paths have the same length, we have $F_1(x) = x_1F_1(e_1)+x_2F_1(e_2)$ and $F_2(x) = x_1F_2(e_1) + x_2F_2(e_2)$. These formulas together with the payoffs in Table 4 show that the internal or restricted game for strategies 1 and 2 has the structure of an anti-coordination game (such as a Hawk-Dove game), which presents an internally neutrally stable (in fact, internally evolutionarily stable) equilibrium $\hat{x}$ where $F_1(\hat{x}) = F_2(\hat{x})$, at $\hat{x}_1 = \frac{20}{37} \approx 0.54$ and $\hat{x}_2 = \frac{17}{37} \approx 0.46$, with $E(\hat{x}, \hat{x}) \approx 1.83$.

Let us check that $\hat{x}$ is path-protecting.

- Strategies that do not get past history $(DD)^4$ when playing with strategies 1 or

2 (they break up or deviate in action before stage 5) obtain at most the minmax payoff $U_{DD} = 0 < E(\hat{x}, \hat{x})$.

- Strategies j that after history $(DD)^4$ play D (as strategy 1 does and strategy 2 does not) may go on generating with 1 and 2 the same paths $a_{11}^{[1,\infty]}$ and $a_{12}^{[1,\infty]}$ as strategy 1 does, may break up at stage 5 (after playing), or may deviate from $a_{11}^{[1,\infty]}$ at stage $T_{j1} > 5$ and from $a_{12}^{[1,\infty]}$ at stage $T_{j2} > 5$, obtaining a payoff (see (6), considering that the pool and population strategy distributions at $\hat{x}$ are the same):

$$F_j(\hat{x}) = \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_j(e_1) + \frac{\hat{x}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_j(e_2).$$

Let us focus first on $F_j(e_2)$ for deviations from $a_{12}^{[1,\infty]}$ after stage 5. Applying lemma D.1, we have that if $F_j(e_2) < F_1(e_2)$ for every possible deviation at $T_{j2} = 6$ (first play of the repeated pattern CC in $h_{12}^{[\infty]}$), then $F_j(e_2) < F_1(e_2)$ for every finite $T_{j2} > 6$. $F_j(e_2)$ for a strategy j that breaks up at $T_{j2} = 5$ or deviates at $T_{j2} = 6$ is bounded by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 5, 5)$, which is $\frac{1-\delta}{1-\delta^6}\delta^4\,(5 + 5\,\delta) = 1.33 < F_1(e_2)$, so $F_j(e_2) < F_1(e_2)$ for $T_{j2} \geq 5$.

Let us focus now on $F_j(e_1)$ for break-up at stage 5 or deviations from $a_{11}^{[1,\infty]}$ after stage 5. The payoff to these strategies is bounded by 0 for $5 \leq T_{j1} \leq 6$ and by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 0, 0, 3, ..., 3, 5)$, which is $\frac{1-\delta}{1-\delta^{T_{j1}}}\left[\frac{\delta^6\,3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\,5\right]$ for $T_{j1} > 6$. Applying lemma D.1 for deviations at $T_{j1} = 7$ (first play of the repeated pattern (CC) in $h_{11}^{[\infty]}$) shows $F_j(e_1) < F_1(e_1) < F_1(e_2)$. We can now state the following bound:

$$F_j(\hat{x}) \leq \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_j(e_1) + \frac{\hat{x}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_1(e_2).$$

Considering that $F_j(e_1) < F_1(e_1) < F_1(e_2)$, and that the weight multiplying $F_1(e_2)$ on the previous convex combination of $F_1(e_2)$ and $F_j(e_1)$ increases with T_{j2}, we find that, for every T_{j1}, the maximum value of the bound corresponds to $T_{j2} = \infty$ (being smaller for finite T_{j2}). Thus, bearing in mind that $F_j(e_1) \leq 0$ for $5 \leq T_{j1} \leq 6$, we have:

$$F_j(\hat{x}) \leq \frac{\hat{x}_2}{\hat{x}_1(1-\delta^5) + \hat{x}_2} F_1(e_2) \approx 1.42 < E(\hat{x}, \hat{x}),\ \textit{for}\ 5 \leq T_{j1} \leq 6.$$

And, for $T_{j1} > 6$,

$$F_j(\hat{x}) \leq \frac{\hat{x}_1(1-\delta)}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2}\left[\frac{\delta^6\,3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\,5\right] + \frac{\hat{x}_2\,F_1(e_2)}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2}.$$

Note that the only variable in the previous bound is T_{j1}, with all the other terms being known numbers. By taking the derivative of this bound with respect to T_{j1} it can be checked that it is monotonic increasing with T_{j1} (for $T_{j1} > 6$), and its limit is $E(\hat{x}, \hat{x})$. Consequently, any strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays D there (as strategy 1 and its path-equivalent-in-$\{1, 2\}$ strategies do) obtains a payoff $F_j(\hat{x}) < E(\hat{x}, \hat{x})$ if j is not path-equivalent to strategy 1 in the set of strategies $\{1, 2\}$.

- We now consider strategies j that after history $(DD)^4$ play C (as strategy 2 does and strategy 1 does not). Applying the same procedure that we followed before, it can be shown that any such strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays C there (as strategy 2 and its path-equivalent-in-$\{1, 2\}$ strategies do), obtains a payoff $F_j(\hat{x}) < E(\hat{x}, \hat{x})$ if j is not path-equivalent to strategy 2 in the set of strategies $\{1, 2\}$.

D Proofs

Proof of proposition 3.1. Let i be a Nash strategy and let $a^{\emptyset} = i(\emptyset)$ be the first action played by i. Let j be a strategy that plays a best-response action to $a^{\emptyset}$ when starting a new partnership, i.e., $j(\emptyset) \in BR(a^{\emptyset})$, and then breaks the partnership. We have $F_j(e_i) = \max_l U(a_l, a^{\emptyset})$. Considering that M is an upper bound for $F_i(e_i)$, the Nash condition $F_i(e_i) \geq F_j(e_i)$ requires $M \geq \max_l U(a_l, a^{\emptyset})$ or, equivalently, $U^{BR}(a^{\emptyset}) \leq M$. □

Proof of proposition 3.2. State x has an associated pool state $p = f^{-1}(x)$. Any strategy arriving at the pool of singles p to be matched faces a distribution of initial actions $q \in \Delta(A)$ (given by $q_k = \sum_{i \in \mathbb{S}(x): i(\emptyset) = a_k} p_i$). Given a state x and its associated q, consider a strategy j that at the beginning of a partnership plays a best response action to the distribution of actions q and then breaks the partnership. The payoff $F_j(x)$ to such a strategy is at least $\underline{m}$. Consequently, if x is Nash, then $E(x, x)$ has to be greater than or equal to $\underline{m}$. For monomorphic states, we have that $F_j(e_i)$ is at least m, while M is an upper bound for $F_i(e_i)$. □

Proof of proposition 3.3. Suppose that i is a strategy with finite self-break-up period T_{ii} and the last action profile $(a_i^{[T_{ii}]}, a_i^{[T_{ii}]})$ in an i-i partnership is not a Nash profile of the stage game G. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage $T_{ii} - 1$, i.e., $j(a_{ii}^{[0,t]}) = i(a_{ii}^{[0,t]})$ for $0 \leq t < T_{ii} - 1$,
- at stage T_{ii} of an i-j partnership plays a best-response action against the action $a_i^{[T_{ii}]}$ played by i at that stage, and
- leaves i (i.e., breaks the partnership with i) after stage $T_{ij} = T_{ii}$.

Strategy j obtains the same stage payoff against i as i itself for the first $T_{ii}-1$ stages of a partnership and a higher payoff at the last stage T_{ii}. Consequently, considering (8), $F_j(e_i) > F_i(e_i)$, so i cannot be a Nash strategy. □

Proof of proposition 3.4. Suppose that i is a Nash strategy with finite self-break-up period T_{ii} that plays the action profile (C, C) at some stage (between stages $t = 1$ and $t = T_{ii}$) of an i-i partnership. Then we have $F_i(e_i) > U(D, D)$. Let t_l be the last stage at which (C, C) is played. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage t_l-1, i.e., $j(a_{ii}^{[0,t]}) = i(a_{ii}^{[0,t]})$ for $0 \leq t < t_l-1$,
- at stage t_l of an i-j partnership plays action D, obtaining a stage payoff $U(D, C) > U(C, C)$, and
- breaks the partnership with i after stage t_l.

  Using formula (8), it can be seen that $F_j(e_i) > F_i(e_i)$, so i is not a Nash strategy (contradiction). The reason, comparing the sequence of payoffs to i in the infinite series $h_{ii}^{[\infty]}$ and to j in the infinite series $h_{ji}^{[\infty]}$ is that j obtains a higher payoff at stage t_l and (if $t_l < T_{ii}$) shortens the sequence of lowest payoffs $U(D, D)$ until the next high payoffs $U(C, C)$ or $U(D, C)$.

□

Proof of proposition 3.5. With the conditions on i, the infinite series of actions that a j-player faces in a population of i-players (see 7) is $(a^N, a^N, ...)$. The best stage-payoff against a^N is obtained by a^N, and, considering that $F_j(e_i) = V(h_{ji}^{\infty}) = (1 - \delta)\sum_{t=1}^{\infty} \delta^{t-1} U(h_{ji}^{[t]})$, the best payoff against any strategy i satisfying the condition is obtained by strategies j that generate the path $h_{ji}^{[\infty]} = ((a^N, a^N), (a^N, a^N), ...)$, which obtain the payoff $F_j(e_i) = U(a^N, a^N)$. If i_1 and i_2 satisfy the conditions for i, we have $F_{i_1}(e_{i_1}) = F_{i_1}(e_{i_2}) = F_{i_2}(e_{i_1}) = F_{i_2}(e_{i_2}) = U(a^N, a^N)$. As $F_i(x)$ is a strictly convex combination of the payoffs $F_i(e_j)$ for $j \in \mathbb{S}(x)$, we have proved the result: if x is a mixture of strategies satisfying the condition for i, we have $F(x, x) = U(a^N, a^N) \geq F_j(x)$ for every $j \in \Omega$.

□

Proof of proposition 3.6. Given any finite set of strategies S, we can number the strategies and identify distributions x whose support is in S with real vectors $\hat{x} \in \Delta(S) \equiv \{\hat{x} \in \mathbb{R}_+^{|S|} : \sum_{k=1}^{|S|} \hat{x}_k = 1\}$. The restriction of F_i to distributions with support in S can then be identified with a function $F_{i|S} : \Delta(S) \to \mathbb{R}$. $F_{i|S}$ is Lipschitz continuous in $\Delta(S)$.[11]

Given any finite set of strategies $S \subset \Omega$ and a neutrally stable state x with support in S, it follows from definition 3 and from the Lipschitz property of the payoff functions $F_{i|S}$ in $\Delta(S)$ that the point $\hat{x} \in \Delta(S)$ associated to state x satisfies the conditions in

[11]See Izquierdo et al. (2021). A function $f : \Delta(S) \to \mathbb{R}$ is Lipschitz continuous in $\Delta(S)$ if there exists a positive real constant K such that, for all x and y in $\Delta(S)$, $|f(x) - f(y)| \leq K||x - y||$.

Thomas (1985) [Theorem 1] to be a weakly evolutionarily stable state in $\Delta(S)$ and, consequently, $\hat{x}$ is Lyapunov stable in the replicator dynamics restricted to S. □

Proof of proposition 3.7. Let $(a^M, a^M) \in A_M$ be one of the symmetric action profiles (there may be more than one) that attain the maximum symmetric payoff $M = \max_{a \in A} U(a,a)$.

Suppose that T_{ii} is finite and $F_i(e_i) < M$. This implies that $h_{ii}^{[\infty]}$ is a repetition of a pattern of length T_{ii}, and, for any fixed t_0, there is always $t > t_0$ with $U(h_{ii}^{[t]}) < M$. Consider a strategy j that when playing with i behaves like i up to period T_{ii}, but at that period does not break the partnership and turns to playing action a^M forever, without breaking the partnership. That would make play between strategy i and strategy j unfold in the same way as it does between two players that play strategy i, with $h_{ji}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$, and hence $F_j(e_i) = F_i(e_i) = F_i(e_j)$. For $t \leq T_{ii}$, two players that play strategy j obtain a payoff $U(h_{jj}^{[t]}) = U(h_{ii}^{[t]}) = U(h_{ij}^{[t]})$. For $t > T_{ii}$, we have $U(h_{jj}^{[t]}) = M$, while $U(h_{ij}^{[t]}) = U(h_{ii}^{[t]}) \leq M$ and, for some $t > T_{ii}$, $U(h_{ij}^{[t]}) < M$. Consequently, considering (8), $F_j(e_j) > F_i(e_j)$, so i is not neutrally stable. Up to now we have proved that if i is neutrally stable with finite T_{ii} then $F_i(e_i) = M$, which implies $U(h_{ii}^{[t]}) = M$ for every t. Suppose that payoff M is obtained at time t_1 by some action profile $h_{ii}^{t_1}$ which is not a Nash equilibrium of the stage game. Then a strategy j that when playing with i chooses the same action as i up to period t_1 (obtaining M at every period up to t_1 if $t_1 \geq 1$), but at period t_1 plays a best response the action taken in $h_{ii}^{t_1}$ and breaks the partnership, obtains a payoff $F_j(e_i) > M = F_i(e_i)$, which cannot happen if i is neutrally stable. □

Proof of proposition 4.1. Let strategy i be weakly path-protecting and let j_1 and j_2 be two alternative best responses to e_i, i.e., $\{j_1, j_2\} \in BR(e_i)$. Considering that the action profiles in $h_{ii}^{[\infty]}$ are symmetric, we have $h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$. In a j_1-j_2 partnership, no strategy can take an action different from the one they take when playing with i until the split-up period $T_{j_1 j_2} = \min(T_{ij_1}, T_{ij_2})$, because the generated histories up to that point are the same as in an i-i partnership and, until they break the partnership, both j_1 and j_2 take the same action as i does given the history. Consequently, $h_{j_1 j_2}^{[\infty]}$ coincides either with $h_{j_1 i}^{[\infty]} = h_{ii}^{[\infty]}$ or with $h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]}$. Then we have $h_{j_1 j_2}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$, which implies that, for $\{j_1, j_2\} \in BR(e_i)$, $F_{j1}(e_{j_2}) = F_{j_1}(e_{j_1}) = F_i(e_i) = F_i(e_{j_1}) = F_i(e_{j_2})$. Now, if y is a mixture of best responses to e_i and j is a best response to e_i, we have $F_j(y) = F_i(e_i) = F_i(y)$ and, consequently, $E(y,y) = F_i(y)$, proving that i is neutrally stable.

□

Proof of proposition 4.2. Let i be a strategy satisfying the conditions of the proposition. It is clear that $T_{ii} = \infty$, $h_{ji}^{[\infty]} = ((\hat{a},\hat{a}),(\hat{a},\hat{a}),...)$ and $F_i(e_i) = U(\hat{a},\hat{a})$. Any strategy j playing with i-players generates a repeated path $h_{ji}^{[\infty]}$ in which the action taken by i is always $\hat{a}$, so, given that $(\hat{a},\hat{a})$ is a (strict) Nash profile and that any deviation from the profile $(\hat{a}, \hat{a})$ is caused by strategy j (i always plays $\hat{a}$, so the second action in the profile is always $\hat{a}$), we have $U(h_{ji}^{[t]}) \leq U(\hat{a}, \hat{a})$ for every t. In fact, since $(\hat{a}, \hat{a})$ is strict Nash, we have $h_{ji}^{[t]} \neq (\hat{a}, \hat{a}) \implies U(h_{ji}^{[t]}) < U(\hat{a}, \hat{a})$, and, considering that $F_j(e_i)$ is a strictly convex combination of the payoffs $U(h_{ji}^{[t]})$, it follows that $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_j(e_i) < U(\hat{a}, \hat{a}) = F_i(e_i)$, proving that i is weakly path-protecting. □

Proof of proposition 4.3. Consider a strategy i such that $T_{ii} = \infty$ and

$$h_{ii}^{[\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^\infty),$$

where:

- Φ_m is a T_m-long repetition of a minmax action profile $(\tilde{a}, \tilde{a})$.
- Φ_f is a T_f-long sequence of symmetric action profiles.
- Φ_p is a T_p-long sequence of symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

As soon as another strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership.

Take Φ_f and Φ_p as fixed, and the length T_m of Φ_m as a parameter. We will show that, for large enough T_m and, then, for large enough δ,

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_j(e_i) < F_i(e_i),$$

i.e., strategy i is weakly path-protecting. By choosing Φ_p in a way such that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, strategy i is also path-protecting.

We will need some intermediate results. First, lemma D.1 implies that, in order to prove the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_j(e_i) < F_i(e_i)$, it is enough to prove that this statement holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful. Consequently, it is enough to consider a finite number of possible deviating paths: those that deviate at some t not greater than $T_m + T_f + T_p$.

Second, the payoff to a strategy that deviates at $t \leq T_m$ is bounded above by the minmax payoff m (because i plays a minmax action up to stage T_m, so the stage payoff for a strategy j at every stage up to and including the deviating stage $t \leq T_m$ is bounded above by m). Let L be the maximum payoff in the stage game. Considering a repeated sequence $(m, ..., m, L, ..., L)$ of T_m payoffs m and $T_f + T_p$ payoffs L, we have that the payoff to a strategy that deviates not later than $T_m + T_f + T_p$ is bounded above[12] by

$$V_L \equiv \frac{m(1-\delta^{T_m}) + \delta^{T_m}(1-\delta^{T_f+T_p})L}{1-\delta^{T_m+T_f+T_p}}.$$

Third, if an infinite sequence of action profiles Φ ends up repeating some finite pattern Φ_1, i.e., if $\Phi = (\Phi_0, (\Phi_1)^\infty)$ for some finite sequences Φ_0 and Φ_1, then[13]

$$\lim_{\delta \to 1} V(\Phi) = \bar{U}_{\Phi_1}.$$

This implies

$$\lim_{\delta \to 1} V_L = \alpha \equiv \frac{m\,T_m + L\,(T_f + T_p)}{T_m + T_f + T_p}, \tag{11}$$

with $\lim_{T_m \to \infty} \alpha = m$, and

$$\lim_{\delta \to 1} F_i(e_i) = \bar{U}_{\Phi_p} > m. \tag{12}$$

Choose some positive $\epsilon < \frac{\bar{U}_{\Phi_p} - m}{3}$. From (11), and considering that α approaches m as T_m grows, we can find a value for T_m such that $\alpha < m + \epsilon$, and then, fixing such T_m, there is some δ_1 such that, for $\delta > \delta_1$, $V_L < m + 2\epsilon$.

From (12), there is some δ_2 such that, for $\delta > \delta_2$, $F_i(e_i) > \bar{U}_{\Phi_p} - \epsilon$. Consequently, for $\delta > \max(\delta_1, \delta_2)$,

$$V_L < F_i(e_i),$$

proving that strategy i is path-protecting.

□

In preparation of the following result, for any finite series of action profiles Φ, let $(\Phi)^k$ represent the sequence made up by repeating k times the action profiles in Φ. Remember that $(\Phi)^\infty$ represents the infinite repetition.

Lemma D.1. *Consider two (not necessarily different) strategies j and i with $h^\infty_{ji} = (\Phi_0, (\Phi_p)^\infty)$, where Φ_0 and Φ_p are finite sequences of action profiles (and where Φ_0 may be empty). Let Φ_1 be another finite sequence of action profiles. If j_1 and j_2 are strategies such that*

$$h^\infty_{j_1 i} = (\Phi_0, \Phi_1)^\infty \text{ and}$$

$$h^\infty_{j_2 i} = (\Phi_0, (\Phi_p)^k, \Phi_1)^\infty \text{ for some } k \in \mathbb{N}$$

then

$$F_{j_1}(e_i) < F_j(e_i) \iff F_{j_2}(e_i) < F_j(e_i).$$

[12]It is easy to check that, for a fixed number of m payoffs T_m, V_L is non-decreasing with the number of L payoffs (V_L is a weighted average of m and $L \geq m$, with the weight of m decreasing if the number of L payoffs increases), so, by taking a number of L values equal to $T_f + T_p$, we can be sure that V_L is an upper bound for the payoff to any strategy that deviates up to $t = T_m + T_f + T_p$.

[13]This can be shown using L'Hopital rule.

Proof of lemma D.1. For any sequence Φ of length $T \geq 1$, let

$$V(\Phi) = \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} U(\Phi^{[t]}).$$

Let the respective lengths of Φ_0, Φ_p and Φ_1 be $T_0 \geq 0, T_p \geq 1$ and $T_1 \geq 1$. If $T_0 = 0$ let $V(\Phi_0) = 0$. Then

$$F_j(e_i) = (1-\delta^{T_0})V(\Phi_0) + \delta^{T_0} V(\Phi_p),$$

$$F_{j_1}(e_i) = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+T_1}}, \text{ and}$$

$$F_{j_2}(e_i) = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{k\,T_p})V(\Phi_p) + \delta^{T_0+k\,T_p}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+k\,T_p+T_1}}.$$

Any of the two conditions $F_{j_1}(e_i) < F_j(e_i)$ or $F_{j_2}(e_i) < F_j(e_i)$ can then be seen to be equivalent (rearranging and simplifying terms) to the condition

$$\delta^{T_1}(1-\delta^{T_0})V(\Phi_0) + (1-\delta^{T_1})V(\Phi_1) < (1-\delta^{T_0+T_1})V(\Phi_p).$$

□

Proof of proposition 5.1. Suppose that a Nash equilibrium state x includes a weakly path protecting strategy i and some strategy j with $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$, then:

- $F_i(x) = E(x, x)$, because x is Nash and i is in its support, so $i \in BR(x)$, and
- $E(x, e_i) < F_i(e_i)$, because i is weakly path-protecting (so it is Nash) and x includes a strategy j that deviates from $h_{ii}^{[\infty]}$ when playing with i, obtaining a payoff $F_j(e_i) < F_i(e_i)$.

Consequently, x is not neutrally stable. □

Proof of proposition 5.2. Let $Eq(x)$ be the set of strategies that are path-equivalent in $\mathbb{S}(x)$ to some strategy in $\mathbb{S}(x)$, and let $\bar{E}q(x)$ be the complement of this set. As x is Nash and path-protecting, we have $F_i(x) = E(x, x)$ for $i \in Eq(x)$ and $F_i(x) < E(x, x)$ for $i \in \bar{E}q(x)$. Consequently, any state y that includes strategies both in $Eq(x)$ (for which $F_i(x) = E(x, x)$) and in $\bar{E}q(x)$ satisfies $E(y, x) < E(x, x)$, and only mixtures of strategies in $Eq(x)$ can be (are) alternative best responses to x. Because any strategy that is path-equivalent in $\mathbb{S}(x)$ to strategy $i \in \mathbb{S}(x)$ behaves like i does with strategies in $Eq(x)$, for any mixture y of strategies in $Eq(x)$ there is an “internal” state $\hat{y}$ satisfying $\mathbb{S}(\hat{y}) = \mathbb{S}(x)$ such that $E(\hat{y}, x) = E(x, x)$, $E(x, y) = E(x, \hat{y})$ and $E(y, y) = E(\hat{y}, \hat{y})$. Consequently, internal neutral stability (which guarantees $E(x, \hat{y}) \geq E(\hat{y}, \hat{y})$) guarantees neutral stability ($E(x, y) \geq E(y, y)$). □

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